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A new type of loop equations

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Abstract: We derive a new form of loop equations for light-like Wilson loops. In bosonic theories those loop equations close only for straight light-like Wilson lines. In the case of $\mathcal{N}=1$ in ten dimensions they close for any light-like Wilson loop. Upon dimensional reduction to $\mathcal{N}=4$ SYM in four dimensions, these loops become exactly the chiral loops which can be evaluated semiclassically, in the strong coupling limit, by a minimal surface in anti de-Sitter space. We show that the AdS calculation satisfies those loop equations. We also find a new fermionic loop equation derived from the gauge theory fermionic equation of motion.

Keywords: 1/N Expansion, String Duality, Conformal Field Models in String Theory.

Contents

1. Introduction

Loop equations were proposed [1] in an attempt to understand the string theory that governs confinement in QCD. The loop equations are dynamical equations for the Wilson loop [2], which in the string picture is the boundary of an open string. The equation is given by a second order differential operator which annihilates smooth loops, and gives a non-trivial result for self-intersecting loops. They can be proven by a formal derivation, and therefore should be solved by any (regularized, but not renormalized) calculation of the Wilson loop. Indeed, they are solved by perturbation theory [3] and by lattice QCD. In the case of two dimensional pure gauge, which is soluble, this solution also solves the loop equation [4].

An important feature of loop equations is that they are correct at all values of the coupling constant. That is the basis for the hope of understanding confinement through them. In particular, they constitute a test for any proposed definition of the theory at strong coupling.

JHEP11(1999)006

The famous Maldacena conjecture [5, 6, 7][1] states that $\mathcal{N} = 4$ supersymmetric U(N) gauge theory in four dimensions is dual to type IIB string theory on a background geometry $AdS_5 \times S^5$ (with certain 5-form flux). According to the conjecture [9, 10] the expectation value of a Wilson loop is given at large N and strong coupling by a minimal surface calculation. This is exactly such a non-perturbative definition of a gauge theory, and it is natural to ask if this ansatz solves the loop equation.

This question was addressed recently [11], and the conclusion was that to the extent that the equations could be checked they are solved by string theory. The main difficulty in applying the loop equations to the AdS calculation was that the AdS ansatz is simple only for a restricted family of loops. If the loop is smooth its expectation value is given by a minimal surface only if it satisfies a local BPS condition that the coupling to the scalars is equal to the coupling to the gauge fields. If one writes the Wilson loop (in lorentzian space) as

$$W = \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \oint \left(A_\mu \dot{x}^\mu + \Phi_i \dot{y}^i \right) ds \right), \tag{1.1}$$

where A_μ are the gauge fields and Φ_i the scalars, the condition is $|\dot{x}| = |\dot{y}|$.

The regular definition of the loop laplacean requires differentiation in all directions, and so breaks this constraint. Therefore in that paper we had to guess how to extend the ansatz also to loops which break the constraint slightly. There is a bigger problem for loops with cusps or intersections, then the constraint is broken by a large amount and therefore the loop equations could not be checked.

In this paper we find a new form for the loop equations which does not require differentiation away from the constraint. One differentiates only in eight independent bosonic and eight fermionic directions which commute with the constraint. Those equations also behave better under supersymmetry, and in fact the loop laplacean is the anticommutator of two fermionic operators. One of the fermionic operators generates another loop equation derived from the gauginos' equations of motion.

Those equations can be derived also in non-supersymmetric gauge theories, but only for a very restricted family of loops. The Wilson loop has to be a straight light-like line.

In the next section we review the regular form of the loop equations (for non-supersymmetric theories).

The derivation of the light-cone loop equations is presented in section 3. The proofs of some of the equations presented in that section are given in the appendix, because they are rather horrible.

We conclude with a discussion on how those loop equations can be applied to the Wilson loops that are calculated by minimal surfaces in AdS. The equations are satisfied by smooth loops with no self intersections. They cannot be checked

[1] for a recent review see [8]

JHEP11(1999)006

when the loops have cusps or intersections, since those loops cannot be calculated by a minimal surface when the constraint is satisfied. We also speculate about other applications for this formalism, and present some suggestions for future work.

2. Short review of loop equations

Let us first review the regular form of the loop equation in pure Yang-Mills, as the new form will use many of the same techniques. For more complete reviews see [12, 13].

The action of pure gauge theory in any number of dimensions is[2]

$$\mathcal{S} = -\frac{1}{4g_{YM}^2}\int dx\ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\,, \tag{2.1}$$

and the Wilson loop is given by

$$W = \frac{1}{N}\,\mathrm{Tr}\ \mathcal{P}\ \exp\left(i\oint A_\mu dx^\mu\right), \tag{2.2}$$

where the integral is over a path parametrized by x^μ.

The first functional derivative of the loop is

$$\frac{\delta}{\delta x^\mu(s)}W = -i\dot{x}^\nu(s)F^a_{\nu\mu}\,\mathrm{Tr}\,\mathcal{P}\left[T^a(s)\exp\left(i\oint A_\mu dx^\mu\right)\right], \tag{2.3}$$

$F_{\mu\nu}$ is the field strength and $T^a(s)$ is a generator of the gauge group inserted at the point s along the loop.

The second derivative can insert another $F_{\mu\nu}$ into the loop, but also has a contact term

$$\begin{aligned}\frac{\delta^2}{\delta x^\mu(s)\delta x_\mu(s')}W =& -\dot{x}^\nu(s)F^a_{\nu\mu}\,\dot{x}^\rho(s')F_\rho{}^{\mu b}\,\mathrm{Tr}\,\mathcal{P}\left[T^a(s)T^b(s')\exp\left(i\oint A_\mu dx^\mu\right)\right] -\\ &- i\dot{x}^\nu(s)D^\mu F^a_{\nu\mu}\delta(s-s')\,\mathrm{Tr}\,\mathcal{P}\left[T^a(s)\exp\left(i\oint A_\mu dx^\mu\right)\right].\end{aligned} \tag{2.4}$$

To write the loop equations we want to retain only the contact term. Therefore we integrate s' over an interval of vanishing width around s, picking out only the delta function term. The loop laplacean is defined as

$$\hat{L} = \lim_{\eta\to 0}\oint ds\int_{s-\eta}^{s+\eta} ds'\frac{\delta^2}{\delta x^\mu(s')\delta x_\mu(s)}\,. \tag{2.5}$$

From (2.4) we get

$$\hat{L}\left\langle W\right\rangle = -i\oint ds\,\dot{x}^\mu\left\langle (D^\nu F_{\mu\nu})^a(s)\,\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\left[T^a(s)\exp\left(i\oint A_\mu dx^\mu\right)\right]\right\rangle. \tag{2.6}$$

[2]The complete action contains a gauge fixing term and ghosts. Those appear also in the equations of motion, but can be dropped by a Ward identity [3].

So we see that $\hat{L}$ inserts the variation of the action $\dot{x}^\mu D^\nu F_{\mu\nu}$ into the loop. This insertion into the loop would be zero if we use the classical equation of motion, but quantum corrections produce contact terms. To see that, one can write the equations of motion as the functional derivative of the action $\mathcal{S}$ and use the Schwinger-Dyson equations, *i.e.* integration by parts in the functional integral,

$$\begin{aligned}\hat{L}\,\langle W\rangle &= g_{YM}^2 \int \mathcal{D}A \oint ds\, \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\left[T^a(s)\exp\left(i\oint A_\mu dx^\mu\right)\right]\dot{x}^\mu(s)\frac{\delta e^{i\mathcal{S}}}{\delta A^{\mu a}(x(s))} \\ &= -g_{YM}^2\left\langle \oint ds\,\dot{x}^\mu(s)\frac{\delta}{\delta A^{\mu a}(x(s))}\frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\left[T^a(s)\exp\left(i\oint A_\mu dx^\mu\right)\right]\right\rangle . \quad (2.7)\end{aligned}$$

The functional derivative $\delta/\delta A_\mu(x(s))$ in this equation is formally evaluated as

$$\begin{aligned}\hat{L}\,\langle W\rangle = -i\frac{\lambda}{N^2}\oint ds \oint ds'\,\delta(x^\mu(s') - x^\mu(s))\dot{x}_\mu(s)\dot{x}^\mu(s') \\ \times\left\langle \mathrm{Tr}\ \mathcal{P}\left[T^a(s)T^a(s')\exp\left(i\oint A_\mu dx^\mu\right)\right]\right\rangle . \qquad (2.8)\end{aligned}$$

We then use the relation between the generators of $\mathrm{SU}(N)$,

$$T^a_{nm}T^a_{kl} = \delta_{nk}\delta_{ml} - \frac{\delta_{nm}\delta_{kl}}{N}\,. \qquad (2.9)$$

Ignoring the $1/N$ term, the trace is broken into two. This gives the correlation function of two loops. In the large N limit, the correlator factorizes and we obtain,

$$\hat{L}\,\langle W\rangle = -i\lambda\oint ds\oint ds'\,\delta(x^\mu(s') - x^\mu(s))\dot{x}_\mu(s)\dot{x}^\mu(s')\langle W_{ss'}\rangle\langle W_{s's}\rangle\,. \qquad (2.10)$$

Here $W_{ss'}$ is a Wilson-loop that start at s and goes to s' and $W_{s's}$ goes from s' to s. They are closed due to the delta function.

According to equation (2.10) $\hat{L}\langle W\rangle$ should receive contributions from self-intersections of the loop. Every intersection point contributes a term proportional to the dot product of the two tangent vectors. Likewise, at a cusp we expect to get a term proportional to the product of the right and left tangent vectors. It seems like we should also count the trivial case of $s = s'$, when $W_{ss'} = 1$ and $W_{s's} = W$. In most of the literature on the loop equation, this trivial self-intersection is ignored. In any case, it can be taken care of by multiplicative renormalization of the loop operator. For a supersymmetric loop, there should be no renormalization and the leading contribution from the trivial self-intersection cancels, because $\dot{x}^2 = 0$.

3. Light-cone loop equations

After presenting the regular loop equations, we derive the light-cone loop equations, utilizing constrained derivatives. First we derive them for bosonic theories (where they usually don't give closed loop equations), and then for the supersymmetric case. Somewhat similar ideas were used in the context of the type IIB matrix model in [14, 15].

3.1 Bosonic case

The bosonic light-cone loop equations can be derived in any dimension. But since we will eventually turn to the $\mathcal{N}=4$ theory in four dimensions, we will do it for the bosonic sector of that theory. For other cases there are changes in numerical factors.

The theory has four gauge fields and six scalars. We work in ten dimensional notations, where they are all denoted by A_μ ($\mu = 0,\ldots,9$). Because euclidean supersymmetry is a bit tricky, we work in lorentzian signature $(-,+,\ldots,+)$.

Our goal is to write loop equations for loops satisfying the on-shell condition $m^2 = \dot{x}_\mu \dot{x}^\mu = 0$. We use light-cone coordinates $\dot{x}_\pm = (\pm\dot{x}_0 + \dot{x}_1)/\sqrt{2}$ so the constraint is $2\dot{x}_-\dot{x}_+ + \dot{x}_i\dot{x}_i = 0$. A variation that preserves this constraint will be of the form $\dot{x}_-\delta\dot{x}_+ + \dot{x}_+\delta\dot{x}_- + \dot{x}_i\delta\dot{x}_i = 0$. We take δx_i to be the independent variations; then one can use reparametrization invariance to set $\dot{x}_-$ to a constant and $\delta\dot{x}_- = 0$. Then solve

$$\delta x_+ = -\int_0^s ds' \frac{\dot{x}_i \delta\dot{x}_i}{\dot{x}_-} = -\frac{\dot{x}_i}{\dot{x}_-}\delta x_i + \int_0^s ds' \frac{\ddot{x}_i \delta x_i}{\dot{x}_-}\,. \tag{3.1}$$

so

$$\frac{\delta x_+(s)}{\delta x_i(s')} = -\frac{\dot{x}_i}{\dot{x}_-}\delta(s-s') + \text{less singular}\,. \tag{3.2}$$

The contribution from the second term in (3.1) is less singular, because of the integral. Since we are interested only in the most singular contact term it can be ignored.

Therefore we can define a differential operator that commutes with the constraint

$$\frac{\delta}{\delta x_i} \sim \frac{\partial}{\partial x_i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+}\,. \tag{3.3}$$

The derivatives on the right hand side are unconstrained functional derivatives.

We define the bosonic light-cone loop laplacean

$$\hat{L}_B^{lc} = \lim_{\eta\to 0} \oint ds \int_{s-\eta}^{s+\eta} ds' \frac{\delta^2}{\delta x_i(s)\delta x_i(s')}\,. \tag{3.4}$$

Naively one would expect that to be[3]

$$\begin{aligned}\hat{L}_B^{lc} &= \left(\frac{\partial}{\partial x_i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+}\right)\left(\frac{\partial}{\partial x_i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+}\right) \\ &= \frac{\partial^2}{\partial x_i \partial x_i} - \frac{\dot{x}_i}{\dot{x}_-}\left(\frac{\partial^2}{\partial x_i \partial x_+} + \frac{\partial^2}{\partial x_+ \partial x_i}\right) + \frac{\dot{x}_i^2}{\dot{x}_-^2}\frac{\partial^2}{\partial x_+ \partial x_+} + \frac{8}{\dot{x}_-}\frac{d}{ds}\left(\frac{\partial}{\partial x_+}\right),\end{aligned} \tag{3.5}$$

where the last term comes from the cross term, $\partial/\partial x_i$ acting on $\dot{x}_i$ (with a factor of 8 from tracing over the index i). But a more careful calculation shows that the

[3] To reduce clutter, we drop the integration signs in this equation, but on the right hand side we still mean that only the contact term is picked out and the expression is integrated around the loop. The same will be done in many of the equations that follow, where some obvious parts of the equations are omitted.

second variation of x_+ is

$$\delta^2 x_+ \sim -\frac{1}{\dot{x}_-}\delta x_i \delta \dot{x}_i = -\frac{1}{2\dot{x}_-}\frac{d}{ds}(\delta x_i)^2 \,. \tag{3.6}$$

The factor of a half on the right hand side means that the cross term should be decreased by a half.

Now we are ready to evaluate the action of this operator on the Wilson loop.

$$\begin{aligned}\hat{L}_B^{lc} &= \frac{\partial^2}{\partial x_i \partial x_i} - \frac{\dot{x}_i}{\dot{x}_-}\left(\frac{\partial^2}{\partial x_i \partial x_+} + \frac{\partial^2}{\partial x_+ \partial x_i}\right) + \frac{\dot{x}_i^2}{\dot{x}_-^2}\frac{\partial^2}{\partial x_+ \partial x_+} + \frac{4}{\dot{x}_-}\frac{d}{ds}\left(\frac{\partial}{\partial x_+}\right) \\ &= -i\dot{x}^\mu D_i F_{\mu i} + i\frac{\dot{x}_i \dot{x}^\mu}{\dot{x}_-}\left(D_- F_{\mu i} + D_i F_{\mu -}\right) - i\frac{\dot{x}_i^2 \dot{x}^\mu}{\dot{x}_-^2} D_- F_{\mu -} - \frac{4i}{\dot{x}_-}\frac{d}{ds}(\dot{x}^\mu F_{\mu -}) \\ &= -i\dot{x}^\mu D^\nu F_{\mu\nu} + i\frac{\dot{x}^\mu \dot{x}^\nu}{\dot{x}_-} D_\nu F_{\mu -} - \frac{4i}{\dot{x}_-}\frac{d}{ds}(\dot{x}^\mu F_{\mu -}) \,. \end{aligned} \tag{3.7}$$

The last term is a total derivative, so for a closed (topologically trivial) loop it integrates to zero.

For a straight line $\ddot{x} = 0$, so the second term is the same as the third and is also a total derivative. Ignoring the total derivatives we find

$$\hat{L}_B^{lc}\langle W\rangle = -\frac{i}{N}\oint ds\, \dot{x}^\mu(s)\left\langle D^\nu F^a_{\mu\nu}(x(s))\, \mathrm{Tr}\,\mathcal{P}\left[T^a(s)\exp\left(i\oint A_\mu \dot{x}^\mu ds\right)\right]\right\rangle . \tag{3.8}$$

This is the same as what the unconstrained loop laplacean gives. Going through the regular procedure (2.6)–(2.10) we get

$$\begin{aligned}\hat{L}_B^{lc}\langle W\rangle &= -i\frac{g_{YM}^2}{N}\oint ds \oint ds'\, \dot{x}^\mu(s)\dot{x}_\mu(s')\delta(x(s) - x(s'))\langle W_1\rangle\langle W_2\rangle \\ &\sim -i\frac{\lambda}{\epsilon^3}\oint ds\, \dot{x}^\mu \dot{x}_\mu \langle W\rangle = 0 + O\left(\frac{\lambda}{\epsilon}\right), \end{aligned} \tag{3.9}$$

plus nonzero contributions at intersections.

The regular loop equations can be defined locally on the loop, not integrated around it. The constrained loop equations work only when integrating along the loop.

If the Wilson loop is not a straight line, the constrained variation does not give a closed loop equation. But the extra term $i\frac{\dot{x}^\nu \dot{x}^\mu}{\dot{x}_-}D_\nu F_{\mu -}$ is canceled in the supersymmetric case.

3.2 Supersymmetric Wilson loop

The $\mathcal{N} = 4$ gauge theory in four dimensions has, in addition to the gauge fields and scalars, also gauginos Ψ. The Wilson loop could couple also to them, or in other words, the bosonic loop we considered so far belongs to a supermultiplet, whose other members have also fermionic parameters. Still working in ten dimensional notations, in the light-cone Ψ breaks into Ψ_1 in the 8_s of SO(8) and Ψ_2 in the 8_c ($\Gamma_-\Psi_1 = \Gamma_+\Psi_2 = 0$).

JHEP11(1999)006

The supersymmetry generators of the gauge theory Q also decompose in the light-cone. Q_1 (after rescaling) acts by

$$\begin{aligned} 2^{1/4}\sqrt{\dot x_-}[Q_{1a},A_\mu] &= \frac{i}{2}\delta^i_\mu\gamma_{ia\dot a}\Psi_{2\dot a}+\frac{i}{\sqrt2}\delta^+_\mu\Psi_{1a}\,,\\ 2^{1/4}\sqrt{\dot x_-}\{Q_{1a},\Psi_{1b}\} &= -\frac14\gamma_{ijba}F_{ij}+\frac12\delta_{ab}F_{-+}\,,\\ 2^{1/4}\sqrt{\dot x_-}\{Q_{1a},\Psi_{2\dot a}\} &= \frac{1}{\sqrt2}\gamma_{ia\dot a}F_{-i}\,, \end{aligned} \tag{3.10}$$

and Q_2 by

$$\begin{aligned} 2^{1/4}\sqrt{\dot x_-}[Q_{1a},A_\mu] &= \frac{i}{2}\delta^i_\mu\gamma^T_{i\dot aa}\Psi_{1a}-\frac{i}{\sqrt2}\delta^-_\mu\Psi_{2\dot a}\,,\\ 2^{1/4}\sqrt{\dot x_-}\{Q_{2\dot a},\Psi_{1a}\} &= -\frac{1}{\sqrt2}\gamma_{i\dot aa}F_{+i}\,,\\ 2^{1/4}\sqrt{\dot x_-}\{Q_{2\dot a},\Psi_{2\dot b}\} &= -\frac14\gamma_{ij\dot b\dot a}F_{ij}-\frac12\delta_{\dot a\dot b}F_{-+}\,. \end{aligned} \tag{3.11}$$

As was shown in [11], the loops satisfying the masslessness condition $\dot x^2=0$ are locally BPS, and depend on only eight fermionic parameters. Therefore, to write the supersymmetric Wilson loop we introduce a fermion ζ_a in the 8_s. The supersymmetric Wilson loop is

$$\begin{aligned} W = \frac1N \operatorname{Tr}\mathcal{P}\Big[&\exp\Big(\oint\zeta(s)(Q_1+\frac12\widetilde Q_1)ds\Big)\exp\Big(i\oint A_\mu\dot x^\mu\,ds\Big)\times\\ &\times\exp\Big(-\oint\zeta(s)(Q_1+\frac12\widetilde Q_1)ds\Big)\Big]\,. \end{aligned} \tag{3.12}$$

$\widetilde Q_1$ (and $\widetilde Q_2$) are other SUSY generators acting on $\dot x^\mu$ giving $\dot\zeta$ by

$$\begin{aligned} \sqrt2\dot x_-[\widetilde Q_{1a},\dot x_\mu] &= -\frac{i}{\sqrt2}\delta^+_\mu\dot\zeta_a\,,\\ \sqrt2\dot x_-[\widetilde Q_{2\dot a},\dot x_\mu] &= -\frac{i}{2}\delta^i_\mu\gamma_{i\dot aa}\dot\zeta_a\,. \end{aligned} \tag{3.13}$$

There are a few justifications for including the action of $\widetilde Q_1$. For one we were unable to write loop equations without it. But also, regarding the Wilson loop as the first quantized action for a w-boson of a broken SU$(N+1)$ gauge group, we should use the supersymmetry generators of that group, which also act on the $\dot x$'s.

It is useful to write the supersymmetric loop as

$$W=\frac1N\operatorname{Tr}\mathcal{P}\Big[\exp\Big(i\oint\mathcal{A}_\mu\dot x^\mu\,ds\Big)\Big]\,, \tag{3.14}$$

where in the exponent we have

$$
\begin{aligned}
i\mathcal{A}_\mu \dot{x}^\mu &= \exp\left(\zeta(Q_1+\frac{1}{2}\widetilde{Q}_1)\right) iA_\mu\dot{x}^\mu \exp\left(-\zeta(Q_1+\frac{1}{2}\widetilde{Q}_1)\right) \\
&= iA_\mu\dot{x}^\mu - \frac{\sqrt{\dot{x}_-}}{2^{\frac{5}{4}}}\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right) + \frac{1}{4\dot{x}_-}A_-\zeta\dot{\zeta} + \\
&\quad + \frac{3}{16\dot{x}_-}\dot{x}_{[i}F_{j-]}\zeta\gamma_{ij}\zeta + \frac{i}{2^{\frac{1}{4}}48\sqrt{\dot{x}_-}}\zeta\gamma_{ij}\zeta\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\zeta\gamma_j\Psi_2 + \cdots
\end{aligned} \tag{3.15}
$$

To simplify the equations we used the anti-symmetric symbol $[\ldots]$ (normalized by 1/6).

The action of $\widetilde{Q}_1$ gives $\frac{1}{4\dot{x}_-}A_-\zeta\dot{\zeta}$. It is natural to absorb $\zeta\dot{\zeta}$ into $\dot{x}_+$, since both couple to A_-. This modifies the on-shell condition to $2\dot{x}_-\dot{x}_+ + \dot{x}_i\dot{x}_i + \frac{i}{2}\zeta\dot{\zeta} = 0$. This way, we can define a constrained and unconstrained fermionic variation, as in the bosonic case. It is

$$
\frac{\delta}{\delta\zeta} \sim \frac{\partial}{\partial\zeta} + \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\,, \tag{3.16}
$$

which we use below.

3.3 Supersymmetric loop equation

The constrained supersymmetric loop laplacean is

$$
\hat{L}^{lc} = \lim_{\eta\to 0}\oint ds\int_{s-\eta}^{s+\eta} ds'\left(\frac{\delta^2}{\delta x_i(s)\delta x_i(s')} - i\frac{\delta}{\delta\zeta(s)}\frac{\overleftrightarrow{d}}{ds}\frac{\delta}{\delta\zeta(s')}\right). \tag{3.17}
$$

The $\zeta = 0$ part of the bosonic variation is still

$$
\hat{L}^{lc}_{B[0]} = -i\dot{x}^\mu D^\nu F_{\mu\nu} + i\frac{\dot{x}^\mu\dot{x}^\nu}{\dot{x}_-}D_\nu F_{\mu-}\,, \tag{3.18}
$$

plus the total derivative.

To this we want to add the fermionic piece. When d/ds acts to the right the $\zeta = 0$ piece is

$$
\begin{aligned}
\frac{\delta}{\delta\zeta}\frac{d}{ds}\frac{\delta}{\delta\zeta}_{[0]} &\sim \left(\frac{\partial}{\partial\zeta} + \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\right)\frac{d}{ds}\left(\frac{\partial}{\partial\zeta} + \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\right) \\
&\sim -\frac{1}{4\sqrt{2}\dot{x}_-}\left\{\left(\sqrt{2}\dot{x}_-\Psi_1 + \dot{x}_i\gamma_i\Psi_2\right), \left(\sqrt{2}\dot{x}_-\Psi_1 + \dot{x}_j\gamma_j\Psi_2\right)\right\} + \\
&\quad + \frac{2\dot{x}^\nu\dot{x}^\mu}{\dot{x}_-}D_\nu F_{\mu-} \\
&\sim -\frac{1}{4}\dot{x}^\mu\left\{\bar{\Psi}, \Gamma_\mu\Psi\right\} + \frac{2\dot{x}^\nu\dot{x}^\mu}{\dot{x}_-}D_\nu F_{\mu-}\,.
\end{aligned} \tag{3.19}
$$

As in (3.6) there is a correction to the second derivative. Again it is equal to minus half the cross term, decreasing it to $\frac{\dot{x}^\nu\dot{x}^\mu}{\dot{x}_-}D_\nu F_{\mu-}$.

When d/ds acts to the left you find only the first term in (3.19). Together this is

$$\hat{L}^{lc}_{F[0]} = -\frac{1}{2}\dot{x}^\mu \left\{\bar{\Psi}, \Gamma_\mu \Psi\right\} + \frac{\dot{x}^\nu \dot{x}^\mu}{\dot{x}_-} D_\nu F_{\mu-}\,. \tag{3.20}$$

Combining this with the bosonic piece gives

$$\begin{aligned}\hat{L}^{lc}_{[0]}\langle W\rangle = &-\frac{i}{N}\oint ds\,\dot{x}^\mu \left\langle \left(D^\nu F^a_{\mu\nu} - \frac{1}{2}\left\{\bar{\Psi}, \Gamma_\mu \Psi\right\}^a\right) \times \right. \\ &\left. \times \operatorname{Tr}\mathcal{P}\left[T^a(s)\exp\left(i\oint \mathcal{A}_\mu \dot{x}^\mu\,ds\right)\right]\right\rangle .\end{aligned} \tag{3.21}$$

This is the equation of motion for the supersymmetric gauge theory, which allows us to complete the loop equations. By the usual manipulations this is equal to

$$\hat{L}^{lc}_{[0]}\langle W\rangle = -i\lambda \oint ds \oint ds'\,\dot{x}^\mu(s)\dot{x}_\mu(s')\delta^4(x(s)-x(s'))\langle W_1\rangle\langle W_2\rangle\,. \tag{3.22}$$

When $\delta/\delta A^\mu$ acts on the bosonic loop it brings down $i\dot{x}_\mu$, but acting on the supersymmetric loop (3.14) will bring down more terms of higher order in ζ. It is easy to see, though, that all those terms are zero when multiplied by $\dot{x}^\mu$.

In the above we considered only the $\zeta = 0$ part of the action of $\hat{L}^{lc}$ on the loop. When $\zeta = 0$ the on-shell constraint is $\dot{x}^2 = 0$, so the right hand side vanishes for a smooth loop. But when $\zeta \neq 0$, as we mentioned before, the constraint is modified to $\dot{x}_\mu \dot{x}^\mu + \frac{i}{2}\zeta\dot{\zeta} = 0$, so we expect to get this extra term at higher orders in ζ. Indeed calculating the term linear in ζ one finds many terms, most of them hopefully cancel, but some terms that remain are

$$\hat{L}^{lc}_{[1]} = -\frac{1}{2^{\frac{1}{4}}\sqrt{\dot{x}_-}}\dot{\zeta}\left(\gamma_i D_i \Psi_2 + \sqrt{2}D_-\Psi_1\right) = -\frac{1}{2^{\frac{1}{4}}\sqrt{\dot{x}_-}}\dot{\zeta}\left(\Gamma^0\Gamma^\mu D_\mu \Psi\right)_1, \tag{3.23}$$

where the right hand side is half of the components of the fermionic equations of motion

$$\frac{\delta \mathcal{S}}{\delta \bar{\Psi}} = -\frac{i}{g^2_{ym}}\Gamma_\mu D^\mu \Psi\,. \tag{3.24}$$

We can use this to write a Schwinger-Dyson equation, or a fermionic loop equation. This will insert into the loop a derivative with respect to Ψ_1

$$\begin{aligned}\hat{L}^{lc}_{[1]}\langle W\rangle &= \frac{1}{N}\oint ds \frac{-\dot{\zeta}(s)}{2^{\frac{1}{4}}\sqrt{\dot{x}_-}}\left\langle \frac{\delta}{\delta \Psi^a_1(x(s))}\operatorname{Tr}\mathcal{P}\left[T^a(s)\exp\left(i\oint \mathcal{A}_\mu \dot{x}^\mu\,ds\right)\right]\right\rangle \\ &= -\lambda \oint ds \oint ds' \frac{1}{2}\dot{\zeta}(s)\zeta(s')\delta^4(x(s)-x(s'))\langle W_1\rangle\langle W_2\rangle\,.\end{aligned} \tag{3.25}$$

Combined with (3.22) we get

$$\hat{L}^{lc}\langle W\rangle = -i\lambda \oint ds \oint ds' \left(\dot{x}^\mu(s)\dot{x}_\mu(s') - \frac{i}{2}\dot{\zeta}(s)\zeta(s')\right)\delta^4(x(s)-x(s'))\langle W_1\rangle\langle W_2\rangle \tag{3.26}$$

JHEP11(1999)006

and the contribution from a smooth point is proportional to $\dot{x}_\mu\dot{x}^\mu - \frac{i}{2}\dot{\zeta}\zeta$, which is indeed zero. If the loop has cusps or intersections there will be a contribution from all those points proportional to $i\lambda/\epsilon^2$ where ϵ is a UV cutoff and a function of the angle between the two tangent vectors.

There might be other terms at higher orders in ζ, but we should hope they are also zero. It would be nice to find a manifestly supersymmetric derivation of those equations, or another way to check them to all orders in ζ.

3.4 Fermionic identities

The SUSY generators of the gauge theory (3.10) and (3.11) have the algebra

$$\begin{aligned}
\{Q_{1a}, Q_{1b}\} &= \frac{i}{2\dot{x}_-}\delta_{ab}D_-\,, \\
\{Q_{1a}, Q_{2\dot{b}}\} &= \frac{i}{2\sqrt{2}\dot{x}_-}\gamma_{ia\dot{b}}D_i\,, \\
\{Q_{2\dot{a}}, Q_{2\dot{b}}\} &= \frac{-i}{2\dot{x}_-}\delta_{\dot{a}\dot{b}}D_+\,.
\end{aligned} \tag{3.27}$$

This algebra is realized on loops by the operators

$$\begin{aligned}
q_1 &= \oint ds\left(\frac{\partial}{\partial\zeta} - \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\right), \\
q_2 &= \frac{1}{\sqrt{2}\dot{x}_-}\oint ds\left(\dot{x}_i\gamma_i q_1 - \frac{i}{2}\frac{\delta}{\delta x_i}\gamma_i\zeta\right) \\
&= \frac{1}{\sqrt{2}\dot{x}_-}\oint ds\left(\dot{x}_i\gamma_i\frac{\partial}{\partial\zeta} - \frac{i}{2}\frac{\partial}{\partial x_i}\gamma_i\zeta + \frac{i}{4\dot{x}_-}\dot{x}_i\gamma_i\zeta\frac{\partial}{\partial x_+}\right).
\end{aligned} \tag{3.28}$$

We prove in the appendix that when acting on a loop $q_1 = Q_1$ and $q_2 = Q_2$ by an explicit calculation for the first few orders in ζ. Part of the calculation is simply based on the construction of the loops in terms of ζQ_1, but at higher orders in ζ the calculation becomes non-trivial involving Fierz identities and loop equations. One can also check the commutators of those operators, it is easy to see that

$$\begin{aligned}
\{q_{1a}, q_{1b}\} &= \frac{-i}{2\dot{x}_-}\delta_{ab}\oint ds\,\frac{\partial}{\partial x_+}\,, \\
\{q_{1a}, q_{2\dot{b}}\} &= \frac{-i}{2\sqrt{2}\dot{x}_-}\gamma_{ia\dot{b}}\oint ds\,\frac{\partial}{\partial x_i}\,, \\
\{q_{2\dot{a}}, q_{2\dot{b}}\} &= \frac{-i}{2\dot{x}_-^2}\delta_{\dot{a}\dot{b}}\oint ds\left(\dot{x}_i\frac{\partial}{\partial x_i} + \dot{x}_+\frac{\partial}{\partial x_+} - \zeta\frac{d}{ds}\frac{\partial}{\partial\zeta}\right) \\
&= \frac{i}{2\dot{x}_-}\delta_{\dot{a}\dot{b}}\oint ds\,\frac{\partial}{\partial x_-}\,,
\end{aligned} \tag{3.29}$$

where the last equality can be easily proven, and is related to reparametrization invariance. Those commutators agree with those of the supercharges up to a sign, as they should.

Note that q_1 is not the same as $\delta/\delta\zeta$ (3.16), rather it has the opposite sign on the second term. This is similar to $\mathcal{N}=1$ supersymmetry in four dimensions where the SUSY generator is the same as the fermionic covariant derivative up to such a sign flip.

Since the action is invariant under supersymmetry, one might have hoped to use the supersymmetry generators to write loop equations. But the supercharges cannot be expressed in terms of constrained variations, therefore this is impossible. Instead the constrained variations are

$$\begin{aligned}\oint ds\frac{\delta}{\delta\zeta} &= q_1 + \oint ds\frac{i\zeta}{2\dot{x}_-}\frac{\partial}{\partial x_+} = Q_1 - \tilde{Q}_1\,,\\ \oint ds\dot{x}_i\gamma_i\frac{\delta}{\delta\zeta} &= q_2 + \oint ds\frac{i\gamma_i\zeta}{2\sqrt{2}\dot{x}_-}\frac{\partial}{\partial x_i} = Q_2 - \tilde{Q}_2\,,\end{aligned} \tag{3.30}$$

but since those are not symmetries of the action, they don't annihilate the loop.

One can also define

$$\begin{aligned}q_3 &= \sqrt{2}\dot{x}_-\oint ds\,\zeta\,,\\ q_4 &= \oint ds\,\dot{x}_i\gamma_i\zeta\,.\end{aligned} \tag{3.31}$$

Among themselves they commute to zero, but with q_1 and q_2

$$\begin{aligned}\{q_{1a},q_{1b}\} &= \sqrt{2}\dot{x}_-\delta_{ab}\,,\\ \{q_{1a},q_{4\dot{b}}\} &= \{q_{2\dot{b}},q_{3a}\} = \dot{x}_i\gamma_{ia\dot{b}}\,,\\ \{q_{2\dot{a}},q_{4\dot{b}}\} &= -\sqrt{2}\dot{x}_+\delta_{\dot{a}\dot{b}}\,,\end{aligned} \tag{3.32}$$

and the right hand side is the light-cone decomposition of $\Gamma^\mu\dot{x}_\mu$.

3.5 Fermionic loop equation

Another very interesting fermionic operator is

$$\hat{D} = \lim_{\eta\to 0}\oint ds\int_{s-\eta}^{s+\eta}ds'\,\gamma_i\frac{\delta}{\delta x_i(s)}\frac{\delta}{\delta\zeta(s')}\,. \tag{3.33}$$

As we show in the appendix, $\hat{D}$ acting on the Wilson loop gives a fermionic loop equation. The $\zeta=0$ piece inserts into the loop

$$\frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\dot{x}^\mu\Gamma_\mu\Gamma_\nu D^\nu\Psi\,. \tag{3.34}$$

Again we use the Schwinger-Dyson equations related to the fermionic equations of motion

$$\frac{\delta\mathcal{S}}{\delta\bar{\Psi}} = -\frac{i}{g_{ym}^2}\Gamma_\mu D^\mu\Psi\,. \tag{3.35}$$

The result is two terms proportional to $\dot{x}_i\gamma_i\zeta$ with a relative minus sign. So if there is no self intersection it's zero.

The piece linear in ζ in $\hat{D}$ gives

$$-\frac{1}{4}\left(D^\mu F_{i\mu} - \frac{1}{2}\left\{\bar{\Psi},\Gamma_i\Psi\right\} - \frac{\dot{x}_i}{\dot{x}_-}\left(D^\mu F_{-\mu} - \frac{1}{2}\left\{\bar{\Psi},\Gamma_-\Psi\right\}\right)\right)\gamma_i\zeta\,. \tag{3.36}$$

This is a linear combination of the bosonic equations of motion. It is nice to see a different linear combination than $\dot{x}^\nu(D^\mu F_{\nu\mu} - \frac{1}{2}\{\bar{\Psi},\Gamma_\nu\Psi\})$ which we got for the supersymmetrized loop laplacean. Again if you continue to the next step you find that each term gives $\dot{x}_i\gamma_i\zeta$, and again at smooth points they are canceled by the relative minus sign.

One might have actually expected two loop equations based on the fermionic equation of motion, but we can get only one. Since $\dot{x}^2 = 0$ the factor $\dot{x}^\mu\Gamma^\mu$ in (3.34) projects onto half the equations. We did get a different fermionic loop equation in the piece linear in ζ in $\hat{L}^{lc}$ (3.23).

Since q_1 and q_2 are symmetries of the loop, anticommuting them with $\hat{D}$ should give an operator which annihilates the loop. The anticommutator with q_1 is trivial, but with q_2 one gets (after some algebra) the light-cone loop laplacean

$$\begin{aligned}\left\{q_{1a},\hat{D}_{\dot{b}}\right\} &= 0\,,\\ \left\{q_{2\dot{a}},\hat{D}_{\dot{b}}\right\} &= \frac{-i}{2\sqrt{2}\dot{x}_-}\delta_{\dot{a}\dot{b}}\hat{L}^{lc}\,.\end{aligned} \tag{3.37}$$

4. Conclusions

Our main result is the operator $\hat{L}^{lc}$ defined in (3.17), which commutes with the constraint $\dot{x}_\mu\dot{x}^\mu + \frac{i}{2}\zeta\dot{\zeta} = 0$ and when acting on a Wilson loop gives a closed loop equation. When the loop is smooth the equation reads

$$\hat{L}^{lc}\langle W\rangle = -i\frac{\lambda}{\epsilon^3}\oint ds\left(\dot{x}^\mu\dot{x}_\mu + \frac{i}{2}\zeta\dot{\zeta}\right)\langle W\rangle \sim 0\,. \tag{4.1}$$

Where $\sim$ means that the most divergent pieces cancel. With cusps or intersections there will be more divergent terms proportional to the product of the two tangent vectors.

There is also a fermionic operator $\hat{D}$, a "Dirac operator" on loop space, which also annihilates the loop. The operator $\hat{L}^{lc}$ is the anticommutator of $\hat{D}$ with a supersymmetry generator.

The AdS/CFT correspondence tells us how to calculate some Wilson loops at strong 't Hooft coupling using minimal surfaces in AdS. The ansatz is

$$\langle W\rangle \sim \exp\left(-\sqrt{\lambda}\tilde{A}\right), \tag{4.2}$$

with $\tilde{A}$ a Legendre transform of the area of the minimal surface ending along the loop at the boundary of AdS.

For smooth loops this calculation, in terms of a minimal surface, is valid only for loops satisfying the constraint $\dot{x}_\mu \dot{x}^\mu = 0$ (where $\dot{x}^\mu$ includes the couplings to both scalars and gauge fields). Therefore we can check if the ansatz solves the light-cone loop equation.

The calculation is very simple. To leading order in λ we get

$$\hat{L}^{lc}\langle W\rangle \sim \lambda \int ds \left(\frac{\delta \tilde{A}}{\delta x^i(s)}\right)^2 \langle W\rangle. \tag{4.3}$$

It was argued [11] that, as long as the constraint is satisfied, the expectation value of the loop is not divergent. Since $\delta/\delta x^i$ commutes with the constraint, it will give a finite result when acting on $\tilde{A}$. That means that there are no divergent terms, as in (4.1).

This is not a very strong test of the conjecture, since those equations are satisfied by any functional that assigns those loops a finite value. But it does agree with the assertion that those loops are finite, and that the leading behavior is proportional to $\sqrt{\lambda}$.

It would be more interesting to check the equation for loops with cusps and intersections, where the right hand side is not zero. Unfortunately, those loops can be evaluated by minimal surfaces only when $\dot{x}^2 \neq 0$. So the new loop equations are not valid. There were problems with using the other form of the loop equations in this case too.

The calculation with the regular loop laplacean required extending the ansatz (4.2) to loops that break the constraint. Applying the light-cone loop laplacean does not require this extension, and is therefore more reliable.

This argument applies to the AdS calculation with both lorentzian and euclidean signature. In the euclidean theory there are still light-like lines, since the parameters that couple to the scalars are imaginary.

We derived the light-cone loop equations for bosonic theories, where they close only for straight light-like lines. There might be applications of those loop equations for QCD. For example, it might be possible to apply them to Wilson loops wrapping the compact light-like direction in DLCQ of gauge theories. For the $\mathcal{N} = 4$ theory they close for any loop satisfying the constraint. We did not check if and when one can write such loop equations for theories with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry, which could be interesting.

When loop equations were first constructed there was a hope to use them to quantize the QCD string. This program was never completed, yet it is tempting to think that if the regular loop equations are related to covariant quantization of the QCD string, the light-cone equations are related to light-cone quantization of the string. Since light-cone quantization is easier in some cases, it might enable progress in this direction. In the case of $\mathcal{N} = 4$ SYM, by the Maldacena conjecture, the QCD string is just a fundamental string in AdS background.

It would be interesting to generalize to arbitrary loops. That would require an operator which commutes with a constraint like $\dot{x}^2 = m^2$ and gives a closed loop equation.

Also, the equations were checked for the few lowest orders in the fermionic parameter ζ. It would be nice to find a formulation that will prove the equations to all orders in ζ.

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A. Filling in some details

A.1 Conventions

We use signature $(-,+,\ldots,+)$ and light cone coordinates

$$\dot{x}^{\pm} = \frac{1}{\sqrt{2}}\left(\pm\dot{x}^0 + \dot{x}^1\right). \tag{A.1}$$

The lagrangean is

$$\mathcal{L} = -\frac{1}{g_{YM}^2}\,\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi\right), \tag{A.2}$$

$D_\mu = \partial_\mu - iA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$.

This lagrangean is invariant under the supersymmetry transformation

$$\begin{aligned} \delta A_\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_\mu\Psi\,, \\ \delta\Psi &= -\frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon\,. \end{aligned} \tag{A.3}$$

The equations of motion are

$$\begin{aligned} \frac{\delta\mathcal{S}}{\delta A^\mu} &= -\frac{1}{g_{ym}^2}\left(D^\nu F_{\mu\nu} - \frac{1}{2}\left\{\bar{\Psi},\,\Gamma_\mu\Psi\right\}\right) = 0\,, \\ \frac{\delta\mathcal{S}}{\delta\bar{\Psi}} &= -\frac{i}{g_{ym}^2}\Gamma^\mu D_\mu\Psi = 0\,. \end{aligned} \tag{A.4}$$

JHEP11(1999)006

We decompose Ψ in the light-cone gauge into two spinors Ψ_1 and Ψ_2 in the 8_s and 8_c of SO(8) respectively. Under this decomposition

$$\Gamma^0\Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i^T & 0 \end{pmatrix} \qquad \Gamma^0\Gamma_+ = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \Gamma^0\Gamma_- = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\Gamma_{ij} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & \gamma_{ij} \end{pmatrix} \qquad \Gamma_{+i} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ -\gamma_i^T & 0 \end{pmatrix} \qquad \Gamma_{-i} = \sqrt{2}\begin{pmatrix} 0 & \gamma_i \\ 0 & 0 \end{pmatrix}$$
$$\Gamma_{+-} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma_{-+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.5}$$

where γ_i are the Clebsch-Gordan coefficients of SO(8).

For example

$$\bar\Psi\Gamma^\mu D_\mu\Psi = \sqrt{2}\Psi_1 D_-\Psi_1 - \sqrt{2}\Psi_2 D_+\Psi_2 + \Psi_1\gamma_i D_i\Psi_2 + \Psi_2\gamma_i^T D_i\Psi_1\,. \tag{A.6}$$

Let us write again what appears in the exponent of the supersymmetric Wilson loop (3.15)

$$\begin{aligned} i\mathcal{A}_\mu\dot{x}^\mu &= iA_\mu\dot{x}^\mu - \frac{\sqrt{\dot{x}_-}}{2^{\frac{5}{4}}}\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right) + \frac{3}{16\dot{x}_-}\dot{x}_{[i}F_{j-]}\zeta\gamma_{ij}\zeta + \\ &\quad + \frac{i}{2^{\frac{1}{4}}48\sqrt{\dot{x}_-}}\zeta\gamma_{ij}\zeta\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\zeta\gamma_j\Psi_2 + \cdots \end{aligned} \tag{A.7}$$

and $\zeta\dot\zeta$ was absorbed into $\dot{x}_+$. Then the loops we are studying satisfy the constraint $2\dot{x}_+\dot{x}_- + \dot{x}_i^2 + \frac{i}{2}\zeta\dot\zeta = 0$. To express that we use the functional variations which commute with the constraint

$$\frac{\delta}{\delta x_i} \sim \frac{\partial}{\partial x_i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+} \qquad \text{and} \qquad \frac{\delta}{\delta\zeta} \sim \frac{\partial}{\partial\zeta} + \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\,, \tag{A.8}$$

where the derivatives on the right hand side are unconstrained functional variations. As was explained in the text, there are corrections to this equation for higher derivatives.

A.2 Evaluating q_1

The first identity is for the supersymmetry generator Q_1 defined in (3.10)

$$Q_1 = q_1 = \frac{\partial}{\partial\zeta} - \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\,. \tag{A.9}$$

To demonstrate this relation we calculate the first few terms in an expansion in ζ

$$Q_{1[0]} = -\frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(\sqrt{2}\dot{x}_-\Psi_1 + \dot{x}_i\gamma_i\Psi_2\right),$$

$$
\begin{aligned}
Q_{1[1]} &= -\frac{1}{4\dot{x}_-}\left(\dot{x}_i F_{j-} + \frac{1}{2}\dot{x}_- F_{ij}\right)\zeta\gamma_{ij} - \frac{1}{4\dot{x}_-}\dot{x}^\mu F_{\mu-}\zeta \\
&= \frac{3}{8\dot{x}_-}\dot{x}_{[i}F_{j-]}\gamma_{ij}\zeta - \frac{1}{4\dot{x}_-}\dot{x}^\mu F_{\mu-}\zeta, \\
Q_{1[2]} &= \frac{i}{2^{\frac{1}{4}}16\sqrt{\dot{x}_-}}\left(\frac{\dot{x}_i}{\dot{x}_-}\gamma_j D_-\Psi_2 + \gamma_i D_j\Psi_2\right)\zeta\gamma_{ij}\zeta \\
&= \frac{i}{2^{\frac{1}{4}}16\sqrt{\dot{x}_-}}\zeta\gamma_{ij}\zeta\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\gamma_j\Psi_2\,.
\end{aligned}
\tag{A.10}
$$

On the other hand

$$
\begin{aligned}
q_{1[0]} &= -\frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(\sqrt{2}\dot{x}_-\Psi_1 + \dot{x}_i\gamma_i\Psi_2\right), \\
q_{1[1]} &= \frac{3}{8\dot{x}_-}\dot{x}_{[i}F_{j-]}\gamma_{ij}\zeta - \frac{1}{4\dot{x}_-}\dot{x}^\mu F_{\mu-}\zeta, \\
q_{1[2]} &= \frac{i}{2^{\frac{1}{4}}48\sqrt{\dot{x}_-}}\left(\zeta\gamma_{ij}\zeta\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\gamma_j\Psi_2 + 2\gamma_{ij}\zeta\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\zeta\gamma_j\Psi_2\right) + \\
&\quad + \frac{i}{2^{\frac{1}{4}}8\sqrt{\dot{x}_-}}\zeta\, D_-\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right) \\
&= \frac{i}{2^{\frac{1}{4}}16\sqrt{\dot{x}_-}}\zeta\gamma_{ij}\zeta\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\gamma_j\Psi_2 + \\
&\quad + \frac{i}{2^{\frac{1}{4}}8\sqrt{\dot{x}_-}}\zeta\left[\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\zeta\gamma_i\Psi_2 + D_-\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right)\right],
\end{aligned}
\tag{A.11}
$$

where we used the Fierz identity $\gamma_{ia\dot{a}}\gamma_{ib\dot{b}} + \gamma_{ia\dot{b}}\gamma_{ib\dot{a}} = 2\delta_{ab}\delta_{\dot{a}\dot{b}}$, which is related by triality to the usual Clifford algebra relation $\gamma_i\gamma_j^T + \gamma_j\gamma_i^T = 2\delta_{ij}$

$$
\gamma_{ij}\zeta\,\zeta\gamma_j\Psi_2 = -\zeta\gamma_i\gamma_j^T\,\zeta\gamma_j\Psi_2 + \zeta\,\zeta\gamma_i\Psi_2 = \zeta\gamma_{ij}\zeta\,\gamma_j\Psi_2 + 3\zeta\,\zeta\gamma_i\Psi_2\,. \tag{A.12}
$$

The last term on the right hand side of (A.11) is zero (for a smooth loop) by the fermionic loop equation

$$
\sqrt{2}\zeta D_-\Psi_1 + \zeta\gamma_i D_i\Psi_2 = \zeta\left(\Gamma^0\Gamma^\mu D_\mu\Psi\right)_1 = \zeta\frac{\delta}{\delta\Psi_1} \sim \zeta\zeta = 0\,. \tag{A.13}
$$

We see therefore that $Q_1 = q_1$.

A.3 Evaluating q_2

Next we show

$$
Q_2 = q_2 = \frac{1}{\sqrt{2}\dot{x}_-}\left(\dot{x}_i\gamma_i q_1 - \frac{i}{2}\frac{\delta}{\delta x_i}\gamma_i\zeta\right). \tag{A.14}
$$

We act with Q_2 (3.11) on the loop and expand

$$
Q_{2[0]} = \frac{1}{2^{5/4}\sqrt{\dot{x}_-}}\left(\sqrt{2}\dot{x}_+\Psi_2 - \dot{x}_i\gamma_i\Psi_1\right),
$$

$$Q_{2[1]} = -\frac{1}{4\sqrt{2}\dot{x}_-}\left(2\dot{x}_- F_{+i} - \dot{x}_j F_{ji} + \dot{x}_i F_{-+}\right)\gamma_i\zeta - \frac{i}{8\sqrt{2}\dot{x}_-}\dot{x}_i F_{jk}\zeta\gamma_{ijk}$$
$$= -\frac{1}{4\sqrt{2}\dot{x}_-}\left(3\dot{x}_{[-}F_{+i]} + \dot{x}^\mu F_{\mu i}\right)\gamma_i\zeta - \frac{1}{8\sqrt{2}\dot{x}_-}\dot{x}_i F_{jk}\zeta\gamma_{ijk}\,. \tag{A.15}$$

On the other hand

$$\frac{1}{\sqrt{2}}\frac{\dot{x}_i}{\dot{x}_-}\gamma_i\, q_{1[0]} = -\frac{1}{2^{\frac{3}{4}}\dot{x}_-^{\frac{3}{2}}}\left(\sqrt{2}\dot{x}_i\gamma_i\dot{x}_-\Psi_1 + \dot{x}_i^2\Psi_2\right)$$
$$= \frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(\sqrt{2}\dot{x}_+\Psi_2 - \dot{x}_i\gamma_i\Psi_1\right),$$
$$\frac{1}{\sqrt{2}}\frac{\dot{x}_i}{\dot{x}_-}\gamma_i\, q_{1[1]} = \frac{1}{8\sqrt{2}\dot{x}_-^2}\left(3\dot{x}_k\dot{x}_{[i}F_{j-]}\gamma_k\gamma_{ij}\zeta - 2\dot{x}_i\dot{x}^\mu F_{\mu-}\gamma_i\zeta\right)$$
$$= -\frac{1}{4\sqrt{2}\dot{x}_-}\left(3\dot{x}_{[i}F_{-+]} - \dot{x}^\mu F_{\mu i} + 2\frac{\dot{x}_i\dot{x}^\mu}{\dot{x}_-}F_{\mu-}\right)\gamma_i\zeta +$$
$$+\frac{1}{8\sqrt{2}\dot{x}_-}\dot{x}_i F_{jk}\gamma_{ijk}\zeta\,, \tag{A.16}$$

and

$$i\frac{\delta}{\delta x_i}\gamma_i\zeta_{[1]} = \dot{x}^\mu F_{\mu i}\gamma_i\zeta - \frac{\dot{x}^\mu\dot{x}_i}{\dot{x}_-}F_{\mu-}\gamma_i\zeta\,. \tag{A.17}$$

Together

$$q_{2[0]} = \frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(\sqrt{2}\dot{x}_+\Psi_2 - \dot{x}_i\gamma_i\Psi_1\right),$$
$$q_{2[1]} = \frac{1}{8\sqrt{2}}\frac{\dot{x}_i}{\dot{x}_-}F_{jk}\gamma_{ijk}\zeta - \frac{1}{4\sqrt{2}\dot{x}_-}\left(3\dot{x}_{[i}F_{-+]} + \dot{x}^\mu F_{\mu i}\right)\gamma_i\zeta\,. \tag{A.18}$$

So $Q_2 = q_2$.

A.4 Evaluating $\hat{D}$

Finally we calculate

$$\hat{D} = \gamma_i\frac{\delta}{\delta x^i}\frac{\delta}{\delta\zeta}\,, \tag{A.19}$$

which gives the fermionic loop equation.

$$\hat{D}_{[0]} = \gamma_i\left(\frac{\partial}{\partial x^i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+}\right)\frac{\partial}{\partial\zeta}_{[0]}$$
$$= -\frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(D_i - \frac{\dot{x}_i}{\dot{x}_-}D_-\right)\gamma_i\left(\sqrt{2}\dot{x}_-\Psi_1 + \dot{x}_j\gamma_j\Psi_2\right)$$
$$= -\frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\Bigg(\sqrt{2}\dot{x}_-\gamma_i D_i\Psi_1 - \sqrt{2}\dot{x}_i\gamma_i D_-\Psi_1 + \dot{x}_j\gamma_i\gamma_j D_i\Psi_2 -$$
$$- 8\frac{d}{ds}\Psi_2 - \frac{\dot{x}_i^2}{\dot{x}_-}D_-\Psi_2\Bigg)$$

$$
\begin{aligned}
&= \frac{1}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(\sqrt{2}\gamma_i\left(\dot{x}_i D_- - \dot{x}_- D_i\right)\Psi_1 + \left(\dot{x}_j\gamma_j\gamma_i D_i + 2\dot{x}_- D_+\right)\Psi_2\right) - \\
&\quad - \frac{1}{2^{\frac{1}{4}}\sqrt{\dot{x}_-}}\dot{x}^\mu D_\mu \Psi_2 \\
&= \frac{\dot{x}^\mu}{2^{\frac{5}{4}}\sqrt{\dot{x}_-}}\left(\Gamma_\mu\Gamma_\nu D^\nu\Psi\right)_2 - \frac{1}{2^{\frac{1}{4}}\sqrt{\dot{x}_-}}\dot{x}^\mu D_\mu\Psi_2 \,. \qquad (A.20)
\end{aligned}
$$

We dropped a total derivative in the course of the calculation. The first term gives a fermionic loop equation

$$
\left(\dot{x}^\mu\Gamma_\mu\Gamma_\nu D^\nu\Psi\right)_2 = \left(\dot{x}^\mu\Gamma_\mu\frac{\delta}{\delta\bar{\Psi}}\right)_2 \propto \dot{x}^\mu\dot{x}^\nu\Gamma_\mu\Gamma_\nu = 0\,, \qquad (A.21)
$$

plus nonzero terms at cusps and intersections. The second term is almost a total derivative, $\dot{x}^\mu D_\mu$ is the bosonic part of d/ds. Since we will look at the term linear in ζ next, we will need

$$
\dot{x}^\mu D_\mu\Psi_2 = \frac{d}{ds}\Psi_2 - \frac{\sqrt{\dot{x}_-}}{2^{\frac{5}{4}}}\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right)\Psi_2 + \cdots \qquad (A.22)
$$

The term linear in ζ contains

$$
\begin{aligned}
&\gamma_i\left(\frac{\partial}{\partial x^i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+}\right)_{[0]}\left(\frac{\partial}{\partial\zeta} + \frac{i\zeta}{4\dot{x}_-}\frac{\partial}{\partial x_+}\right)_{[1]} = \\
&\quad = \frac{1}{8\dot{x}_-}\left(2\dot{x}_j D_i F_{k-} - 2\frac{\dot{x}_i\dot{x}_j}{\dot{x}_-}D_- F_{k-} + \dot{x}_- D_i F_{jk} - \dot{x}_i D_- F_{jk}\right)\gamma_i\gamma_{jk}\zeta + \\
&\qquad + \frac{1}{4\dot{x}_-}\left(\dot{x}^\mu D_i F_{\mu-} - \frac{\dot{x}^\mu\dot{x}_i}{\dot{x}_-}D_- F_{\mu-}\right)\gamma_i\zeta - \frac{2}{\dot{x}_-}\frac{d}{ds}F_{i-}\gamma_i\zeta \\
&\quad = -\frac{1}{4}\left(D^\mu F_{i\mu} - \frac{\dot{x}_i}{\dot{x}_-}D^\mu F_{-\mu}\right)\gamma_i\zeta\,. \qquad (A.23)
\end{aligned}
$$

There are two more contributions at this order with two Ψ's and a ζ. One was mentioned above, the other comes from the fermionic part of $\delta/\delta x^i$

$$
\begin{aligned}
&\gamma_i\left(\frac{\partial}{\partial x^i} - \frac{\dot{x}_i}{\dot{x}_-}\frac{\partial}{\partial x_+}\right)_{[1]}\frac{\partial}{\partial\zeta}_{[0]} + \frac{1}{2\sqrt{2}}\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right)\Psi_2 = \\
&\quad = -\frac{1}{4\sqrt{2}}\left(\left(\zeta\gamma_i\Psi_2\right)\gamma_i\left(\sqrt{2}\Psi_1 + \frac{\dot{x}_j}{\dot{x}_-}\gamma_j\Psi_2\right) - 2\left(\sqrt{2}\zeta\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\zeta\gamma_i\Psi_2\right)\Psi_2\right) \\
&\quad = \frac{1}{4\sqrt{2}}\gamma_i\zeta\left(\sqrt{2}\Psi_2\gamma_i\Psi_1 + \frac{\dot{x}_i}{\dot{x}_-}\Psi_2\Psi_2\right), \qquad (A.24)
\end{aligned}
$$

by a Fierz identity. In those equations an anticommutator of the Ψ's is implied, which eliminates terms like $\Psi_2\gamma_{ij}\Psi_2$.

Together (A.23) and (A.24) give

$$\gamma_i \frac{\delta}{\delta x^i} \frac{\delta}{\delta \zeta}_{[1]} = -\frac{\gamma_i \zeta}{4} \left(D^\mu F_{i\mu} - \Psi_2 \gamma_i \Psi_1 - \frac{\dot{x}_i}{\dot{x}_-} D^\mu F_{-\mu} - \frac{1}{\sqrt{2}} \frac{\dot{x}_i}{\dot{x}_-} \Psi_2 \Psi_2 \right). \tag{A.25}$$

This is a linear combination of the bosonic equations of motion which vanishes, by a loop equation, at a smooth point.

References

[1] Y.M. Makeenko and A.A. Migdal, *Exact equation for the loop average in multicolor QCD, Phys. Lett.* **B 88** (1979) 135; *Quantum chromodynamics as dynamics of loops, Nucl. Phys.* **B 188** (1981) 269.

[2] K.G. Wilson, *Confinement of quarks, Phys. Rev.* **D 10** (1974) 2445.

[3] R.A. Brandt, A. Gocksch, M.A. Sato and F. Neri, *Loop Space, Phys. Rev.* **D 26** (1982) 3611.

[4] V.A. Kazakov and I.K. Kostov, *Nonlinear strings in two-dimensional* U(∞) *gauge theory, Nucl. Phys.* **B 176** (1980) 199

[5] J. Maldacena, *The large-N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys.* **2** (1997) 231 [hep-th/9711200].

[6] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory, Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[7] E. Witten, *Anti-de Sitter space and holography, Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[8] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity,* hep-th/9905111.

[9] J. Maldacena, *Wilson loops in large-N field theories, Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[10] S-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity,* hep-th/9803001.

[11] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces,* hep-th/9904191.

[12] A.A. Migdal, *Loop equations and 1/N expansion, Phys. Rep.* **102** (1984) 199.

[13] A.M. Polyakov, *Gauge fields and strings,* Harwood Academic Publishers, Chur 1987.

[14] M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, *String field theory from IIB Matrix Model, Nucl. Phys.* **B 510** (1998) 158 [hep-th/9705128].

[15] K. Hamada, *Supersymmetric Wilson Loops in IIB Matrix Model, Phys. Rev.* **D 56** (1997) 7503 [hep-th/9706187].

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